ECONOMIC GROWTH AND CAPITAL ACCUMULATION

1. *From Adam Smith to Arthur Lewis.*

"The design of the book is different from that of any treatise on Political Economy which has been produced in England since the work of Adam Smith." "The last great book covering this wide range was John Stuart Mill's *Principles of Political Economy.*" The first sentence is from Mill's preface, the second from the preface to Lewis' *The Theory of Economic Growth.* It would be rash to conclude from this sequence that one might keep up-to-date in economics by reading a new book every century. Lewis' remark is partly a warning that his book is about applications as well as theories, and partly a reminder that he is taking up an old theme of English economic thought. When Keynes solved "the great puzzle of Effective Demand", he made it possible for economists once more to study the progress of society in long-run classical terms—with a clear conscience, "safely ensconced in a Ricardian world."

The aim of this paper is to illustrate with two diagrams a theme common to Adam Smith, Mill, and Lewis, the theory of which is perhaps best seen in Ricardo: namely, the connexion between capital accumulation and the growth of the productive labour force. The neo-classical economists were in favour of productivity and thrift, but never found a way to make much use of them. Earlier views were much more specific: for example, Adam Smith's industry "proportioned to capital", Ricardo's Doctrine of Unbalanced Growth, Mill's "Irish peasantry, only half fed and half employed", now so familiar in the work of Harrod, Nurkse, or Lewis, and in a hundred United Nations reports. Nevertheless, our illustration takes a neo-classical form, and enjoys the neo-classical as well as the Ricardian vice.[1]

2. *An Unclassical Case.*

In the first instance, capital and labour are the only factors of production. In a given state of the arts, the annual output Y depends on the stock of capital K and the labour force N, according to the constant-elasticity production function $Y = K^{\alpha} N^{\beta}$. With constant returns to scale, $\alpha + \beta = 1$. The annual additon to the capital stock is

1. An appendix discusses some of the questions—especially those raised by Joan Robinson—concerning the role of Capital as a factor of production in the neo-classical theory. However, the appendix makes no attempt to discuss or defend the use of this or other concepts in a dynamic analysis, except by indicating some very artificial assumptions by which the main difficulties might be dodged.

the amount saved[2] sY, where s is a given ratio of saving to output (or income.)

Therefore the annual relative rate of growth of capital is $s\frac{Y}{K}$. The symbols y and n stand for the annual relative rates of growth of output and labour respectively. In these terms the production function implies the basic formula for the rate of growth of output[3]:

$$(1)\ y = \alpha s\frac{Y}{K} + \beta n$$

Effective demand is so regulated (*via* the rate of interest or otherwise) that all savings are profitably invested, productive capacity is fully utilized, and the level of employment can never be increased merely by raising the level of spending. The forces of perfect competition drive the rate of profit or interest r and the (real) wage rate w into equality with the marginal productivities of capital and labour, derived from the production function:

$$(2)\quad r = \alpha\frac{Y}{K} \quad (3)\quad w = \beta\frac{Y}{N}$$

Thus the profit rate is proportional to output per unit of capital, $\frac{Y}{K}$, or the *output-capital ratio;* the wage rate is proportional to output per unit of labour, $\frac{Y}{N}$, or *output per head.* The relative shares of total profits and total wages in income are constants, given by the production elasticities α and β.

In Figure 1, look first at the three heavy lines. That rate of growth of capital $s\frac{Y}{K}$ is shown as a function of the output-capital ratio by a line through the origin with a slope equal to the saving ratio ($s = 10$ per cent).[4]

This may be called the *growth line* of capital. The resulting contribution of capital to the growth of output, $\alpha s\frac{Y}{K}$, is another line through the origin, of slope αs ($\alpha = 0{\cdot}4$), and may be called the *contribution line* of capital.[5]

2. A given amount of saving, in terms of output, has a constant productive equivalent in terms of the capital stock. In Joan Robinson's language, the Wicksell effect is assumed to be zero. Part IV of the appendix argues that Joan Robinson is mistaken in her view that a rule can be laid down regarding the direction of the Wicksell effect.

3. The formula is obtained after logarithmic differentiation of the production function. All variables are treated as continuous functions of time, which is measured in years. For example, "annual output" is the instantaneous rate of output per annum. The words "growth", "rate of growth", etc. always refer to instantaneous relative rates of growth per annum, subject to instantaneous compounding.

4. Numerical plottings are used merely to help fix ideas.

5. According to the marginal productivity formula (2) above, $\alpha s\frac{Y}{K} = sr$. The rate of profit may therefore be read directly from Figure 1 by multiplying the contribution line of capital by 10.

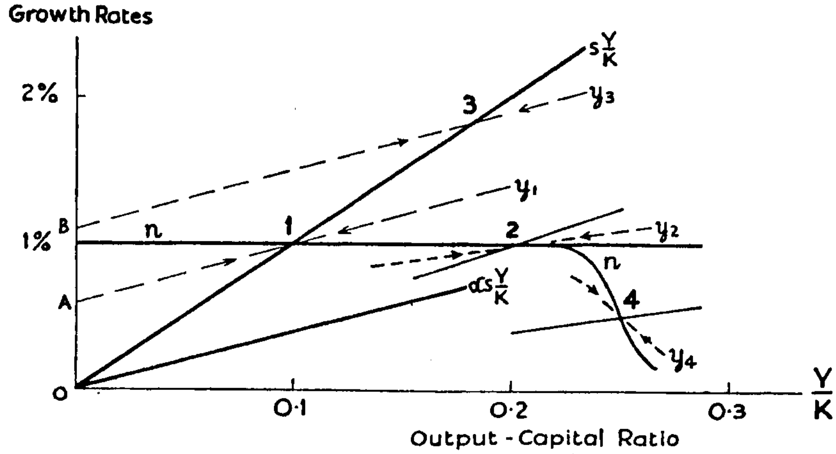

FIGURE 1.

The growth line of labour is horizontal, the rate of growth of the labour force being assumed for the present to be constant ($n = 1$ per cent). The distance OA on the vertical axis is βn ($\beta = 1 - \alpha = 0{\cdot}6$), which is labour's contribution to the growth of output. Adding the contributions of capital and labour gives the growth line of output, y_1. Since $\alpha + \beta = 1$, the geometry of the diagram implies that the three growth lines (of capital, labour, and output) must intersect at the same point (1), where growth in each case is 1 per cent per annum. The growth line of output is the intermediate between the growth lines of labour and capital, and divides the vertical distance between them in the proportion $\alpha : \beta$. Anywhere west of (1) output is growing faster than capital, so the output-capital ratio is rising—moving eastward. Anywhere east of (1), capital is growing faster than output, so the movement of the output-capital ratio is westward. Only at (1) is there a resting-place. *At any other point the economy is always in motion towards (1)*,[6] as shown by the arrows on the line y_1.

The point (2) is another equilibrium point like (1), except that it corresponds with a saving ratio of 5 per cent, instead of 10 per cent at (1). The (unlabelled) continuous line is the new growth line of capital, with a slope through the origin of 5 per cent. The line y_2 is the new growth line of output, which if extended would meet the vertical axis at A as before. (The new contribution line of capital is not drawn.) At (2) economic growth is uniformly 1 per cent, just as at (1), because the three growth lines must still intersect somewhere on the horizontal line n. The given rate of growth of labour thus determines the equilibrium growth rate of the whole economy, while the saving ratio determines the output-capital ratio at which equilibrium will occur.

Suppose the economy is at (2), and that a thrift campaign suddenly raises the saving ratio from 5 per cent to 10 per cent. The growth line of output shifts from y_2 to y_1. Output per head begins to improve (as shown by the height of y_1 above n near (2)), and the wage rate rises in the same proportion. The output-capital ratio gradually sinks westward, and the profit rate sinks in the same proportion. The improvement of output per head continues at an ever-slackening pace down the slope of y_1, towards (1). At (1) output per head and the wage rate are higher than at (2), while the output-capital ratio and the profit rate are lower. These are permanent changes, but the rate of economic growth is faster only in the course of transition from (2) to (1).

Suppose next that the state of the arts, hitherto assumed constant, continually improves. "Neutral" technical progress contributes to the growth of output an annual m per cent beyond the contributions of capital and labour. In Figure 1 the distance AB on the vertical axis is m, at an assumed rate of $\frac{1}{2}$ per cent. The new growth line of output y_3 shows this amount added on top of y_1 (for a 10 per cent saving ratio), and it cuts the growth line of capital at the point (3). This will now be the equilibrium point. In some respects the transition from (2) to (1) is reversed by the introduction of technical progress, since the output-capital ratio and the rate of profit are both higher at (3) than at (1). But the main change is that output per head is not only permanently higher, but perpetually rising (as shown by the height of y_3 above n at (3)). Its rate of increase is actually greater than the m per cent contributed directly by technical progress, because the contribution of capital is also sustained by technical progress at a higher level (as shown by the height of y_1 above n at (3)).

The effect of a change in thrift, assuming constant technical progress, is not shown in the diagram. If another situation were depicted, combining the 5 per cent saving ratio of (2) with the

6. Figure 1 is in effect the "phase portrait" of a first-order differential equation in the variable $\frac{Y}{K}$ (cf. Andronow & Chaikin, *Theory of Oscillations,* Chapter IV). A similar device is used by R. M. Solow (*A Contribution to the Theory of Economic Growth,* Quarterly Journal of Economics, February, 1956). The approach to the point (1) along y_1 is asymptotic—i.e., (1) is reached only in infinity, the speed of travel towards it being directly related to the distance remaining to be travelled. (The time taken for $\frac{Y}{K}$ to move any given distance towards equilibrium is not revealed by the diagram, and can be discovered only by integrating the differential equation.) A point such as (1) is a *stable* equilibrium point because the growth line of capital cuts the growth line of labour (and so the growth line of output) from below. If over a certain range the saving ratio s were a decreasing function of $\frac{Y}{K}$, the growth line of capital might cut the growth line of labour from above, and this second intersection would be an *unstable* equilibrium point (the arrows would be directed away from it on either side).

½ per cent technical progress of (3), its equilibrium point would be found to lie well to the east of (3)', but on exactly the same parallel of economic growth. So long as technical progress and the rate of growth of labour are taken as *data*, they jointly determine the equilibrium growth rate of output and capital.[7] After a transitional phase, the influence of the saving ratio on the rate of growth is ultimately absorbed by a compensating change in the output-capital ratio.

This conclusion is not really surprising. It is in fact the counterpart in our present unclassical model of the classical proposition that capital accumulation leads *ultimately* to the stationary state. A rise in the saving ratio does mean that the level of output per head is permanently higher at any time thereafter than it would have been otherwise. Further, the "transitional phase" is never literally completed; the "transitional" acceleration-deceleration of growth might be visible for centuries, depending entirely on the numerical assumptions. However, only extreme assumptions could produce such a result. It is at first sight disconcerting to find that "plausible" figuring suggests that even the impact effect of a sharp rise in the saving ratio may be of minor importance for the rate of growth: for example, the maximum amount added to the rate of growth, at the beginning of the transition from (2) to (1) in Figure 1, is only 0·4 per cent, though the thrift campaign doubles the saving ratio at a point where the yield on capital is 8 per cent.

To this anti-accumulation, pro-technology line of argument there are at least two possible anwers. First, the rate of technical progress may not be independent of the rate of accumulation, or (what comes to much the same thing) accumulation may give rise to external economies, so that the true social yield of capital is greater than any "plausible" figure based on common private experience.[8] This point would have appealed to Adam Smith, but it will not be pursued here. Second, the rate of growth of labour may

7. Equation (1) on page 335, with the addition of m per cent technical progress, becomes

$$(1') \quad y = \alpha s \frac{Y}{K} + \beta n + m$$

from which it follows that when $y = s\frac{Y}{K}$ (i.e., at an equilibrium point where the output-capital ratio is stationary) $y = \frac{\beta n + m}{1 - \alpha}$. For $\alpha + \beta = 1$ this is simply $y = n + \frac{m}{1 - \alpha}$.

8. In Figure 1, allowing for external economies, $\alpha + \beta$ would exceed unity and so y_1 would cut the growth line of capital above the level of n, just as y_3 does. If the external economies were concentrated on the side of capital (rather than labour), this elevation would take the form of a steeper slope for the contribution line of capital, which of course would no longer correspond with the rate of profit.

not be independent of the rate of accumulation. This is the distinctively classical answer.

In Figure 1 the sloping branch of the growth line of labour represents a situation in which the supply of labour is "elastic" in the vicinity of a certain level of output per head (and wage rate). This situation may be given a Malthusian interpretation, as the response of population to an improvement in the means of subsistence; it may be a situation of "disguised unemployment", with unproductive labour kept in reserve (by sharing with relatives, etc.) at a minimum living standard; it may be the result of Trade Union resistance or some other kind of institutional or conventional barrier, expressed in real terms; or it may reflect a potential supply of migrant labour, available if satisfactory living standards are offered. In any of these situations, "demand for commodities is not demand for labour" (if only Mill had understood his own doctrine): the growth, or productive employment, of the labour force depends directly on the rate of accumulation. In the neighbourhood of the point (4), which is drawn for a saving ratio of 2 per cent, a higher saving ratio will evidently raise the rate of economic growth almost in proportion—and not only "transitionally", but in equilibrium as well. On the other hand, the wage rate and output per head (of *productively employed* labour) will not be much improved; nor will the rate of profit and the output-capital ratio suffer much decline.

This last fact is of course one of the reasons why capital accumulation appears so much more effective in raising the rate of economic growth when faster growth means primarily a faster expansion of productive employment, rather than a faster improvement of output per head. But the main reason is that accumulation is justly credited with the productive contribution of the additional labour that it "sets in motion."

It is now possible to look at Figure 1 in a new light. What is the maximum rate of labour growth consistent with the maintenance of a given standard of output per head? The answer (assuming no technical progress) is that for any such standard—i.e., at any given level of the output-capital ratio—the maximum rate of growth is directly proportional to the saving ratio. In fact, the growth line of capital $s\frac{Y}{K}$, wherever it lies, is the locus of all growth rates at which output per head is constant.

This is a more classical view of the problem, and also, unfortunately perhaps more relevant to many contemporary problems of population pressure and economic growth. However, to see its implications in either context, it is necessary to introduce a characteristic feature of the classical model—namely, the limited "powers of the soil".

3. *A Classical Case.*

A fixed factor of production, which may be called land, can be introduced very simply. Let its production elasticity be γ. Then, assuming constant returns to scale, $\alpha + \beta + \gamma = 1$. However, since land is fixed in supply, it does not appear in the basic formula for the growth of output, $y = \alpha s \frac{Y}{K} + \beta n$, but makes its presence felt by reducing the sum of α and β below unity. With this interpretation, the former marginal productivity relationships for r and w remain unchanged, and there is now a third, of similar form, to determine the rent of land. α, β, and γ are now the constant relative shares of the three factors in income.

If Figure 1 were drawn for $\alpha + \beta < 1$, the growth line of output would cut the growth line of capital below the horizontal growth line of labour. The only possible equilibrium with constant labour growth (and no technical progress) would be one in which output per head and the wage rate were perpetually falling. However, an answer can be given to the question: what rate of labour growth will maintain constant output per head? This condition can be expressed by putting $y = n$ in the formula repeated in the last paragraph, which gives $n = \frac{\alpha}{1 - \beta} s \frac{Y}{K}$. In Figure 2 the constants are assumed to be $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}5$, $\gamma = 0{\cdot}2$ and $s = 10\%$. The coincident values of y and n that satisfy the condition of constant output per head are shown as a function of the output-capital ratio by the growth line $n_1 = y_1$, and this *locus classicus* is called the Ricardian line.

Along the Ricardian line labour is growing as fast as is compatible with a given living standard—keeping pace with the growth of output. Except at the origin, the growth line of capital lies above the Ricardian line, for capital must always grow faster than labour in order to sustain output per head in the face of continually diminishing returns on the land. But since capital is therefore also growing faster than output, the output-capital ratio is continually falling; the profit rate is falling in the same proportion, the wage rate is stable, while rent per acre is rising in proportion with output. As the output-capital ratio falls, the growth rate of labour and output gradually recedes down the slope of the Ricardian line, retreating from the unequal struggle against niggardly nature. In this manner the natural progress of society continues indefinitely towards the origin, where at last the growth line of capital and the Ricardian line intersect, at the point (1), in a stationary state.

If "long indeed before this period the very low rate of profit" has "arrested all accumulation", the change will have been seen in Figure 2 in the form of a decline in the saving ratio, reducing the

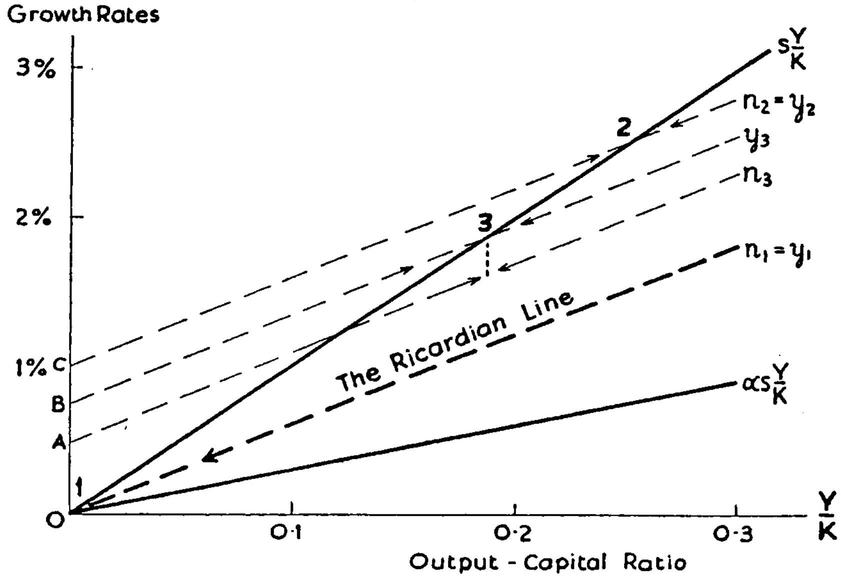

FIGURE 2.

slopes of the growth lines down to zero. On the other hand, a higher saving ratio would proportionally raise the growth rates at every point, but only temporarily interrupt the inevitable progress towards stationariness.

Suppose that this "gravitation . . . is happily checked at repeated intervals" by a constant rate of technical progress ($m = \frac{1}{2}$ per cent, as before). The new version of the Ricardian line $n_2 = y_2$ will lie above the old by the distance OC, which is $\frac{m}{1-\beta}$.[9] At the point (2) where the new line intersects the growth line of capital, technical progress is exactly balanced against diminishing returns, and output per head is constant with a growth rate of about $2\frac{1}{2}$ per cent. So instead of gravitating towards the origin, the economy if necessary levitates to this stable equilibrium point. Ricardo would no doubt object that if population is supposed to grow for ever at $2\frac{1}{2}$ per cent, it is very likely that at some point diminishing returns will set in with a violence not allowed for in our production function.

Given the proposition that at *some* standard of living (for the purpose of the foregoing argument it may be low or high) population will multiply fully in proportion with output, there is perhaps something to be said for the classical "law" of historically diminishing returns. It is in relation to the Malthusian postulate that the classical

9. This is obtained by putting $y = n$ in $y = \alpha s\frac{Y}{K} + \beta n + m$.

vision failed most signally. Suppose then that each generation demands samething better for its children: population is regulated so as to achieve, not a given living standard, but a progressive annual improvement of q per cent. The new element q affects the growth line of labour (now the locus of all labour growth rates consistent with q per cent improvement in output per head) exactly as if q were subtracted from the rate of technical progress, m.[10] Assuming q to be as little as $\frac{1}{4}$ per cent, the new growth lines n_3 and y_3 determine an equilibrium growth rate of labour of about $1\frac{1}{2}$ per cent at the point (3).

Growth lines such as n_1, n_2, and n_3 can be considered, not as time-paths which on various improbable assumptions the economy would follow towards an equilibrium point, but rather as a grid that divides the economic map into characteristic zones of improvement or determination in output per head. Every point on the map represents a particular conjunction of a labour growth rate with an output-capital ratio. In Figure 2, any point to the south and east of n_1 is a situation in which output per head is rising even if there is no technical progress. Between n_1 and n_3 output per head in the absence of technical progress would be falling, but with the assumed $\frac{1}{2}$ per cent technical progress it is rising by more than $\frac{1}{4}$ per cent. Between n_3 and n_2 the rise is less than $\frac{1}{4}$ per cent. A labour growth rate that strays to the north and west of n_2 incurs a decline in output per head, unless technical progress is greater than $\frac{1}{2}$ per cent.

A higher saving ratio, even though it does not change the growth rate at any of the so-called equilibrium points, swings the whole grid to the north and west. As a result, there is a larger area of desirable situations to the south and east of any given criterion of improvement, and a smaller area of undesirable situations to the north and west.

4. *The Harrod Model*

The model used above differs from Harrod's model of economic growth only in that it systematizes the relations between the "warranted" and "natural" rates of growth, and introduces land as a fixed factor.

Any point on the growth line of capital, $s\frac{Y}{K}$, is Harrod's warranted rate of growth $G_w = \frac{s}{C_r}$, since the output-capital ratio $\frac{Y}{K}$ at any given level is the reciprocal of Harrod's capital coefficient C_r. The corresponding point on the growth line of output is Harrod's natural rate of growth G_n. At an equilibrium point, where the two growth lines intersect, the warranted and natural rates of growth are equal.

10. The obtained by putting $y = n + q$ in the formula of the last footnote.

At any other point the wage rate and the profit rate are moving in such a way as to induce entrepreneurs to adjust the output-capital ratio in the direction which will bring the warranted and natural rates of growth together. Specifically, a reduction in the output-capital ratio (an increase in Harrod's C_r) always involves a decline in the rate of profit, and this automatically implies the appropriate movement of the wage rate.

Harrod envisages exactly the same mechanism of adjustment, *via* the "deepening" factor, d, which "may have a positive value because the rate of interest is falling". He argues that natural market forces cannot be expected to achieve the desired results, but does not despair that Keynesian policies may be successful. Nevertheless some of his readers seem to have been misled into the belief that in Harrod's model equality between the warranted and the natural rates of growth can occur only "by a fluke".[11] Harrod's own view is stated very clearly:

"Our aim should be to get such a progressive reduction in the rate of interest that $G_w\, C_r = s - d = G_n\, C_r$. If d is positive, C_r will increase through time, and may eventually become so great as to enable us to dispense with d. At that point interest need fall no further."[12]

The mechanism of Figures 1 and 2 merely makes explicit what this statement implies.

T. W. SWAN

Australian National University,
Canberra.

APPENDIX: NOTES ON CAPITAL[1]

I Joan Robinson's Puzzle.
II The Wicksell Effect.
III Åkerman's Problem.
IV The Wicksell Effect in Reverse.

I. *Joan Robinson's Puzzle*

If we had to put up a scarecrow (as Joan Robinson calls it) to keep off the index-number birds and Joan Robinson herself, it would look something like this: Labour and Land are homogeneous man-

11. Joan Robinson, *The Accumulation of Capital*, p. 405.
12. *Towards A Dynamic Economics*, p. 96.

1. These notes are concerned with certain difficulties in the idea of Capital as a factor of production that were first seen by Wicksell and have now been greatly elaborated by Joan Robinson in two articles (*Review of Economic Studies* 1953-54, and *Economic Journal* 1955) and in her book, *The Accumulation of Capital.* See also the articles by D. G. Champernowne and R. F. Kahn, and by D. G. Champernowne, in the same number of *R.E.S.*, and by R. M. Solow in *R.E.S.* 1955-56. The criticism here ventured is in no sense a book review: it touches on only one aspect of a very important book, and in relation to the text above is best regarded as a face-saving gesture.

hours and acres respectively; Capital is made up of a large number of identical meccano sets, which never wear out and can be put together, taken apart, and reassembled with negligible cost or delay in a great variety of models so as to work with various combinations of Labour and Land, to produce various products, and to incorporate the latest technical innovations illustrated in successive issues of the Instruction Book; Output consists of goods (including meccano sets) that are all produced and sold at constant price-ratios amongst themselves, no matter how the rates of wages, rents, and profits may vary—e.g. they are all produced by similar (but continuously variable) combinations of Labour, Land, and Capital, with similar efficiency, and under similar competitive conditions; Saving = Investment = Accumulation is the current output of meccano sets, and can always be measured (by virtues of the constant price-ratios) at a constant value per meccano set in terms of peanuts or any other consumption product forgone: etc. With assumptions of this kind, the basic model of the text could be rigorously established in a form that would deceive nobody.

Fortunately, economists have usually been willing to hope that even very complicated aggregates like Output might somehow still contain, for some purposes, a rough kernel of meaning "in an index-number sense"—a meaning not literally dependent upon the fantastic assumptions required to avoid all index-number ambiguities. Joan Robinson[2] has spoilt this game for us by insisting that the social Capital, considered as a factor of production accumulated by saving, cannot be given *any* operative meaning—not even in the abstract conditions of a stationary state.

2. The following passages are from the first of Joan Robinson's articles (*R.E.S.* 1953-54, pp. 81 and 82):

"The student of economic theory is taught to write $O = f(L, C)$ where L is a quantity of labour, C a quantity of capital and O a rate of output of commodities. He is instructed to assume all workers alike, and to measure L in man-hours of labour; he is told something about the index-number problem involved in choosing a unit of output; and then he is hurried on to the next question, in the hope that he will forget to ask in what units C is measured. Before ever he does ask, he has become a professor, and so sloppy habits of thought are handed on from one generation to the next. . . .

When we are discussing accumulation, it is natural to think of capital as measured in terms of product. The process of accumulation consists in refraining from consuming current output in order to add to the stock of wealth. But when we consider what addition to productive resources a given amount of accumulation makes, we must measure capital in labour units, for the addition to the stock of productive equipment made by adding an increment of capital depends upon how much work is done in constructing it, not upon the cost, in terms of final product, of an hour's labour. Thus, as we move from one point on a production function to another, measuring capital in terms of product, we have to know the product-wage rate in order to see the effect upon production of changing the ratio of capital to labour. Or if we measure in labour units, we have to know the product-wage in order to see how much accumulation would be required to produce a given increment of capital. But the wage rate alters with the ratio of the factors: one symbol, C, cannot stand both for a quantity of product and a quantity of labour time."

That there should be great difficulties in handling the concept of Capital in a process of change is not surprising. A piece of durable equipment or a pipe-line of work-in-progress has dimensions in time that bind together sequences of inputs and outputs jointly-demanded or jointly-supplied at different dates.[3] The aggregation of capital into a single stock at a point of time is thus the correlative of an aggregation of the whole economic process, not only in cross-section (which gives rise to the ordinary index-number problems), but also in time itself: in other words, the reduction of a very high-order system of lagged equations—in which each event, its past origins and its future consequences, could be properly dated and traced backward and forward in time—to a more manageable system with fewer lags. This second kind of aggregation introduces a further set of ambiguities, similar in principle to those of index-numbers, but as yet hardly investigated.[4] Our scarecrow assumptions dodge both sets of ambiguities—the first because all price-ratios within Output are held constant, the second because Capital, in the form of meccano sets, is both infinitely durable and instantaneously adaptable. This is an extreme of aggregation. From the idea of capital as a single stock there is in principle no sudden transition to "the enormous who's who of all the goods in existence"[5]. Between the two extremes lies an ascending scale of nth-order dynamic systems, in which capital like everything else is more and more finely subdivided and dated, with ascending degrees of (potential) realism and (actual) complexity. In fact, most of us are left at ground-level, on ground that moves under our feet.

In a stationary state, all the complexities of dating disappear. Related inputs and outputs, investments and returns, events and expectations, may be generations removed, but what happens at any time

3. Like the wool and mutton of Marshall's sheep (Wicksell, *Lectures,* Vol. 1, p. 260).

4. By what test should the (relative) success of a proposed scheme of aggregation be judged? Probably, by some measure of the degree of preservation of the dominant behaviour patterns which would be represented in the linear case by the larger roots of an appropriate high-order micro-system; surely not by the degree of "realism" of the scarecrow assumptions necessary to give literal validity to the low-order macro-system. In the present context, we are saved the trouble of trying to apply the suggested criterion by the fact that no appropriate micro-system is available (no one has yet set *Value and Capital* in motion, except under assumptions almost as rigidly fantastic as our own scarecrow). The puzzle may actually be easier if there are strong non-linear features (floors, ceilings, thresholds, etc.), since these sometimes lead to a limited number of highly charactertistic patterns of behaviour.

If the hands of a clock give a fair approximation to the even flow of time, in spite of the diminishing force exerted by the spring as it unwinds, our thanks are due to the discontinuous, non-linear mechanism of the escapement. This result is achieved at the price of an ambiguity—the finite intervals between ticks. Even the best clock is a mere scarecrow model of Time, and absurdly unrealistic from the viewpoint of the General Theory of Relativity. A bad clock that still ticks, or a good one wrongly set, may mean that its user misses the bus.

5. Joan Robinson, *R.E.S.* 1953-54, pp. 83-85.

is exactly repeated at any other time. No information is lost and no ambiguity created by aggregating all times into an eternal present; the shadows of past and future appear only in the form of profit steadily accruing at the ruling rate of interest on the time-consuming investment processes of which every moment has its share. Joan Robinson still insists that Capital cannot be given a meaning conformable with those neo-classical exercises in comparative statics for which the stationary state is expressly designed. In particular, Capital cannot be put into a production function from which, given the supplies of Labour and Land, under perfect-competition profit-maximization assumptions, the equilibrium rates of real wages, rent, and profit may be deduced in the form of marginal productivities. If this scheme is unworkable in a stationary state, it can hardly be sensible to retain it in a dynamic model (like that of our text).

From the various accounts given by Joan Robinson it is not easy to pick out the "basic fallacy" of the marginal productivity scheme. In the passage already quoted[6] she makes the novel suggestion that the production function itself works only if the stock of capital is measured by its value in wage-units, in which case it becomes useless for explaining the equilibrium factor-rewards. But it soon appears that, whatever may be the defects of the neo-classical production function, one geared to what Champernowne calls J.R. units can produce only mental and diagrammatic contortions. (Are there perhaps signs in *The Accumulation of Capital* of the reluctant beginnings of regret for this aberration?)

Frequently Joan Robinson seems to be explaining the factor-rewards by a widow's cruse type of distribution theory.[7] At first sight, it is not altogether clear whether she puts this forward as an independent explanation, or is merely exhibiting the other side of the double-entry national income accounts, which of course always balance in exact confirmation of the theory.[8] Before long, the reader begins

6. p. 344, note 2.

7. The widow's cruse theory of distribution is set out by Nicholas Kaldor (who calls it the Keynesian theory) in *R.E.S.* 1955-56. p. 94 ff. Briefly, the theory says that, given the ratio of investment to income, and given the propensities to save out of profits and wages respectively, the distribution of income between profits and wages must be such as to make the saving ratio equal the investment ratio. For examples see *The Accumulation of Capital* pp. 48, 75-83, 255, 271, 312, 331.

8. Consider the following example of widow's cruse reasoning (*The Accumulation of Capital*, p. 312):

> "The relation of the rate of profit to the marginal product of investment is seen in its simplest form in the imagined state of bliss, where the highest technique known is already in operation throughout the economy and population is constant. The marginal product of investment is then zero. If there is no consumption out of profit (and no saving out of wages or rents) the rate of profit also is zero, and the wages and rent bill absorbs the whole annual output.
>
> But if there is consumption out of profits (and no saving out of rent or wages) the rate of profit remains positive, for the prices of commodities (in

to understand that this profoundly arithmetical organon is not a rival economic calculus, but a subsidiary device that applies, as it were, only in blank spots on the map of economic calculation.[9] In Joan Robinson's world, the blanks are enlarged into great zones marked out by frontier-lines of technical discontinuity. Typically, all products are consumed in fixed proportions and capable of being produced by a single discontinuous "hierarchy of techniques"; each technique has its own fixed factor-proportions, which are rigidly unadaptable; techniques displace each other in profitability across iso-cost "frontiers" defined by well-separated critical sets of factor-rewards; any frontier applies uniformly and simultaneously to every industry throughout the economy.[10] Only at the frontier between two techniques is economic calculation, or the Principle of Substitution, fully effective. Within each zone (i.e. within the limits of the critical sets of factor-rewards along its frontiers) almost anything may happen.

The paradoxes and fabulous histories that enliven *The Accumulation of Capital* have their licence in these extremes of discontinuity. They are not the consequence of any special feature of Joan Robinson's view of capital or of the marginal calculus. When eventually

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relation to money wages and rents) are such that their total selling value exceeds their total costs by the amount of expenditure out of profits. The total of real wages then falls short of total output by the amount of consumption of rentiers, that is, of purchase of commodities out of profit and rent incomes."

It is possible to clarify the two cases distinguished in these paragraphs. In a state of bliss with constant population investment is zero. Since therefore the amount saved has to be zero, any positive profit-saving propensity, whether unity or less, certainly precludes a positive amount of profits; so the second paragraph must refer only to a zero profit-saving propensity, as in effect its two sentences twice affirm (profits = selling value — (wages + rents) = consumption out of profits). Keeping the assumption of no saving out of wages or rents, the meaning of Joan Robinson's two cases may now be rendered as follows:

(i) If the capitalists save some part of any profits they may get, they get no profit, *either* because the rate of profit on capital employed is zero *or* because their saving propensity "multiplies" output itself down to zero (bliss becomes Nirvana).

(ii) If the capitalists consume the whole of any profits they may get, their profits equal the amount they consume, which is positive if they get any profits to consume. (The reason for this ham and eggs dictum is supplied by Kaldor, *R.E.S.* 1955-56, p. 95.)

The marginal productivity theory, on the other hand, infers that in a state of bliss the rate of profit is zero: capital has become a free good, and full employment output is blissfully consistent with zero profits. The true contribution of Kenesian theory is to point out that in these circumstances any positive saving propensity *out of wages or rents* is inconsistent with full employment output.

9. Just as the multiplier theory (to which the widow's cruse is closely related) applies when aggregative economic calculation is suspended by Keynesian unemployment.

10. Note how "realistic" it seems to allow for technical discontinuities (one thinks of coke-ovens, blast furnaces, etc.). The end-result in Joan Robinson's model is that the opportunities for substitution are limited in any situation to a single, universal choice between two techniques; and this restriction becomes the dominant feature of the economy as a whole. The "neo-classical vice of implicit theorising" has here its counterpart in the vice of *explicit realism*.

she considers a primitive agricultural economy in which continuous factor substitution is for once allowed,[11] her theory runs familiarly in terms of discounted products, following Wicksell. Again, when she indulges in some pronouncements concerning the nature of the factor of production and their marginal products, the substance of her thoughts is recognizable as that of the so-called Austrian view of the rôle of capital and time in production, more especially in the form which it was given by Wicksell.[12]

To find the kernel of Joan Robinson's meaning, it is best to go back to Wicksell,[13] to whose ideas she pays repeated and generous tribute. Part II of this appendix examines the problem from Wicksell's viewpoint. But first look again at the early Joan Robinson polemic against the neo-classical scheme (quoted on p. 344), and consider four comments:

1. The value of a stock of capital "in terms of product" is no more plausible than its value in J.R. units as an input to be fed into a production function. If Capital is to be treated, from a productive viewpoint, on all fours with Labour and Land, it must somehow be measured "in terms of its own technical unit",[14] in spite of the obvious difficulties. (In our scarecrow model, it is the *number* of meccano sets, along with the number of manhours and acres, that directly determines the volume of Output—not their value in terms of anything, although it happens that in our model the value of a meccano set in terms of product is constant.)

Joan Robinson is correct in so far as she is complaining that the neo-classical tradition contains no indication of how a "technical unit" for capital may be devised.

11. *The Accumulation of Capital*, pp. 291-292. "This . . . is purely a repetition of our former argument in a simpler setting."

12. *The Accumulation of Capital*, pp. 310-311. Cf. Wicksell, *Lectures*, p. 150.

13. If I were advising a student on a method of approach to *The Accumulation of Capital*, I would recommend the following preparations: (1) Re-read the whole of Wicksell's *Lectures*, Vol. 1. (2) Concentrate in particular on a full understanding of the discontinuous "exception" described by Wicksell in the last complete paragraph on page 177; this sets the stage for Joan Robinson's book. (3) Read the two articles by D. M. Bensusan-Butt, *Oxford Economic Papers*, September, 1954, and *American Economic Review*, September, 1954, for an analysis which in several respects anticipates Joan Robinson, but in terms of a much simpler, and also more flexible, model.

14. Wicksell, *Lectures*, p. 149. Wicksell rejected the possibility of employing such a unit, partly because he could see no way of combining the different kinds of "tools, machinery, and materials, etc." in a "unified treatment" (though he did not shrink from treating labour, land, and product as if each were homogeneous), and partly because he believed at the time that the Walrasian solution of the pricing of newly-produced capital goods involved "arguing in a circle". This latter problem, he thought, would still have to be solved before the "yield" of particular capital goods could be linked up with the rate of interest. Later (cf. *Lectures*, p. 226) he seems to have realised that this criticism was mistaken. The circularity that worried Wicksell (the fact that the rate of interest enters as a cost in the production of capital goods themselves) is merely another aspect of the mutual inter-dependence of all variables in the Walrasian system. For a similar example, see p. 357.

2. On the other hand, there is an ambiguity in her view that capital "measured in terms of product" goes naturally with a discussion of accumulation because "accumulation consists in refraining from consuming current output in order to add to the stock of wealth". What is the addition to the stock of wealth that is made by current accumulation? It is not the *change in the value* of the stock, measured in terms of product, but rather the *value of the change*, measured in terms of product, i.e. the (real) value of the current output not consumed, and so added to the stock. Only the latter element corresponds with the idea of accumulation, unless current output, income, saving, and investment are defined to include current revaluations of the pre-existing stock of capital goods—which is not the conventional usage.[15]

Joan Robinson apparently takes it for granted that the value of the capital stock in terms of product is the same thing as the cumulated value (the time-integral) of investment and saving in terms of product. The two measures may in fact diverge very widely, if there is any change in relative values between "capital goods" and "product": in the first measure, every such change is reflected in an immediate revaluation of the whole capital stock; in the second, the capital stock is recorded as a "perpetual inventory" accumulated by saving at original cost in terms of product. The first measure is certainly Wicksell's. The second is the neo-classical tradition—or rather the tradition of all the economists from Adam Smith to Keynes who have thought of current output as being divided between consumption on the one hand and additions to the capital stock (saving, investment, accumulation) on the other. In a stationary equilibrium, the two measures coincide: but they part company even for infinitesimal variations at the margin of a stationary state, if those variations involve any change in relative values between "capital goods" and "product."

3. A measure of the capital stock in its own technical unit—if that were feasible, as with meccano sets—is of course not the same thing as either of the two measures "in terms of product". When capital is measured in meccano sets, its marginal productivity—the rate of increase of product with respect to the number of meccano sets employed—does not correspond in equilibrium with the ruling rate of profit (the rate of interest), but with that rate multiplied

15. The value changes that accrue as "depreciation" are of course allowed for in the conventional definitions, but we are here concerned with quite a different issue. The relevant distinction is often made explicit in national income statistics in the form of an *inventory revaluation adjustment*. This adjustment is made only in respect of inventories, since revaluations of other capital assets are not included in the figures in the first place.

by the value of a meccano set in terms of product.[16] At the same time, the rate of increase in the number of meccano sets is the current rate of saving divided by the value of a meccano set. Thus for the purpose of any marginal calculation in which the value of a meccano set enters as a constant, Capital may be measured in terms of product (as the accumulation of saving), and its marginal productivity will then correspond with the rate of profit.

This is the *rationale* of the neo-classical procedure. Two elements of calculation are involved, and in both the value of a meccano set is correctly taken as constant: (a) a maximization process in which all prices, including the price of meccano sets, are treated in perfect competition as constant parameters, and (b) a marginal increment of accumulation—the translation of a small amount of product by saving into additional meccano sets—in which the "error" in measuring the capital stock arising from an associated marginal change in the value of a meccano set is confined to the marginal addition being made to the capital stock, and so is of "the second order of smalls". (The revaluation of the pre-existing stock, which occurs in the measure of capital in terms of product that Joan Robinson has in mind, is of a very different order.)

As soon as this point is accepted, it follows that the neo-classical procedure does not, after all, depend on the existence of a "technical unit" of capital—the meccano set—the value of which is in any case cancelled out. For marginal variations about the stationary equilibrium position—i.e. for all the purposes of the neo-classical theory—the natural unit of Capital is simply "an equilibrium dollar's worth" regardless of the physical variety of capital goods, and regardless of marginal value-changes or marginal adaptations of capital towards different physical forms.[17]

4. The foregoing argument is evidently quite symmetrical with respect to every factor and indeed every product: at the margin of a stationary state, Capital, Labour, Land, and Output can each be measured in terms of "an equilibrium dollar's worth". From the "tangency" and "convexity" conditions prevailing at the equilibrium point—by virtue of the first and second-order conditions of the economic maximization process—all valid theorems (as Samuelson might say) can then be deduced. In the given unit, aggregation is

16. To the profit-maximising, cost-minimising entrepreneur, the cost of the annual services of a meccano set, comparable with the real wage in terms of product which is the cost of the annual services of a unit of labour, is the annual interest bill on the price of a meccano set in terms of product.

17. This is essentially the familiar principle which Samuelson calls the Wong-Viner-Harrod envelope theorem (P. A. Samuelson, *The Foundation of Economic Analysis*, pp. 34, 66n, 243; also in *R.E.S.* 1953-54, p. 5). See also Marshall (rebuking J. A. Hobson), *Principles*, p. 409n.

itself quite superficial: the equalities and inequalities that hold for the aggregates at the equilibrium point hold uniformly in terms of "an equilibrium dollar's worth" of every possible subdivision of factor and product right down to the level of the individual firm. That is why the neo-classical theory often appears to be both aggregative and exact.

However, this achievement involves an inherent limitation, against which Joan Robinson has all along been tilting. The theory tells us something about the properties of an equilibrium point, but it gives no information in finite terms about one point in relation to any other point.[18] For instance, it does not enable us to "draw" the hypothetical isoquants of a production function combining (say) Capital and Labour. All we know, from the neo-classical or any similar theory, is certain curvative properties that must hold at any point that is capable of being an equilibrium point. Assuming one such point, we are entitled to draw an invisibly small segment of a curve with the known properties—a grin without a cat. Yet why should we expect a theory to produce even a hypothetical cat? The trouble is that if we were supplied with all the hypotheses or empirical data in the world, we should still be puzzled to draw the rest of the curve, because we should want each point on it to be a potential equilibrium point, whereas our unit—"an equilibrium dollar's worth"—is defined only for a single equilibrium point, and changes its character at any point separated by a finite distance from the first. It may do no harm to sketch in a metaphorical curve, provided the argument touches only a single equilibrium point and its immediate neighbourhood. On these terms, "comparative statics" is a misnomer: not different situations, but only "virtual" displacements at the margin of one situation, can be considered.

For structural comparisons "in the large" (e.g. as between two stationary states with different factor endowments), either the variables must be measurable in naturally homogeneous technical units (like meccano sets and manhours), or else some artificial means must be found to co-ordinate measurements made at different points. For the latter purpose, Champernowne has proposed the use of a *chain index*,[19] an approach which is entirely in keeping with the true character of aggregative analysis. Champernowne's chain index, as presented, looks like a rather *ad hoc* and specialized device to cope with Joan Robinson's difficulties. The next part of these notes is intended to show how a chain index of capital emerges naturally from the analysis of a simple problem considered by Wicksell.

18. In a footnote (*The Accumulation of Capital,* p. 414n.) Joan Robinson says something rather like this.
19. *R.E.S.* 1953-54, p. 112.

II. *The Wicksell Effect*

Joan Robinson finds extraordinary significance in Wicksell's demonstration that an increase in the social capital is partly "absorbed by increased wages (and rent), so that only the residue . . . is really effective as far as a rise in production is concerned".[20] "The amount of employment offered by a given value of capital depends upon the real-wage rate. At a lower wage rate there is a smaller value of a given type of machine."[21] To its discoverer, the *Wicksell effect* seemed mainly important as an obstacle to the acceptance of "von Thünen's thesis", the marginal productivity theory of interest. To Joan Robinson, "this point of Wicksell's is the key to the whole theory of accumulation and of the determination of wages and profits".[22]

To identify the Wicksell effect we may re-work very briefly his "point-input, point-output" case.[23] For Wicksell's grape-juice we substitute labour, imagining a productive process in which the application of an amount of labour N at a point of time results after a "period of production" t in a final output Q which is greater the longer the period of production allowed. Other variables are the real wage rate w, the value of capital in the form of goods-in-process K (both w and K measured in terms of product), and the competitive rate of interest or profit r. Interest is instantaneously compounded; e is the base of natural logarithms. Wicksell's main equations then appear as follows:

$$(1)\quad Q = Nf(t) \qquad (2)\quad w = \frac{Q}{N}e^{-rt} \qquad (3)\quad r = \frac{f'(t)}{f(t)}$$

$$(4a)\quad K = Nw\int_0^t e^{rx}\,dx = Nw\,\frac{e^{rt}-1}{r}$$

$$(4b)\quad K = \frac{Q - Nw}{r}$$

Equation (1) is the production function, showing output per unit of labour as an increasing function $f(t)$ of the period of production. Equations (2) and (3) flow from (1) under perfect-competition profit-maximization assumptions: the wage rate is the discounted product per unit of labour, and the interest rate the (relative) "marginal productivity of waiting". Equation (4a) evaluates K from the cost side as the wages bill continuously invested in production and

20. *Lectures*, p. 268.
21. *The Accumulation of Capital*, p. 391.
22. *The Accumulation of Capital*, p. 396.
23. *Lectures*, pp. 172-181, and in particular pp. 178-180. In the following re-formulation, nothing material is changed. The reader is referred to Wicksell for further explanations. Where our symbols differ from his, the equivalents are: $\frac{Q}{N} = W, N = V_0, r = \rho$. Wicksell's "one hectolitre of grape-juice", which does not appear explicitly, is our N, but the change is only formal. Our numbering of the equations is not the same as Wicksell's. For the mathematics of this case, see R. G. D. Allen, *Mathematical Analysis for Economists*, pp. 248, 362, 403.

cumulated at compound interest over the period t; while (4b), using (1) and (2), shows that K is also the capitalized value of total profits $Q - Nw$. Using the second-order maximization condition upon $f(t)$, Wicksell proved that in equilibrium (for given N) increasing K necessarily means increasing w, decreasing r, and increasing t. In Joan Robinson's language, increasing t represents a higher "degree of mechanization", made profitable as w increases—the *Ricardesque effect*.

Differentiating (1) and (4a) or (4b) while holding N constant, Wicksell next derived a formula for the rate of increase of Q with respect to K. This formula can be expressed in four distinct ways:

$$\text{(5a)} \quad \frac{dQ}{dK} = r - K\left(\frac{r}{w}\frac{dw}{dK} + \left(\frac{rt}{1-e^{-rt}} - 1\right)\frac{dr}{dK}\right)$$

$$\text{(5b)} \quad = r + \left(K\frac{dr}{dK} + N\frac{dw}{dK}\right)$$

$$\text{(5c)} \quad = r + (K - Nwt)\frac{dr}{dK}$$

$$\text{(5d)} \quad = r - \left(\frac{K - Nwt}{wt}\right)\frac{dw}{dK}$$

The second term in these expressions is always negative—i.e. the "marginal productivity of capital" in this sense is always less than the rate of interest, part of the increase in K having been "unproductively absorbed". This is the Wicksell effect. Our four versions suggest that it is somewhat misleading to ascribe the "absorption" simply to increased wages. Version (a) shows the Wicksell effect from the viewpoint of the *cost* of the capital stock (4a), as the consequence of a higher wage rate only partly offset by a lower interest rate. Version (b), from the viewpoint of *capitalized profits* (4b), shows it as the consequence of a lower interest rate only partly offset by a higher wage rate. Versions (c) and (d) use the relation between r and w given in (2) to attribute the whole Wicksell effect on the one hand to a lower interest rate, and on the other hand to a higher wage rate. The multiplicity of explanations shows how treacherous is the idea of causation amongst interdependent variables.

Yet the different versions of the Wicksell effect have one common feature which is itself a complete explanation. To see this, we first write out the logarithmic total differential of (4a) and (4b), giving four alternative expressions for a (proprotional) change in the value of capital to parallel the four versions of (5):

$$\text{(6a)} \quad \frac{dK}{K} = \left[\frac{dN}{N} + \frac{rt}{1-e^{-rt}}\frac{dt}{t}\right] + \left[\frac{dw}{w} + \left(\frac{rt}{1-e^{-rt}} - 1\right)\frac{dr}{r}\right]$$

$$(6b) \qquad = \left[\frac{Q}{Kr}\frac{dQ}{Q} - \frac{Nw}{Kr}\frac{dN}{N}\right] - \left[\frac{Nw}{Kr}\frac{dw}{w} + \frac{dr}{r}\right]$$

$$(6c) \qquad = \left[\quad " \qquad\qquad " \quad\right] - \left[\frac{K - Nwt}{K}\frac{dr}{r}\right]$$

$$(6d) \qquad = \left[\quad " \qquad\qquad " \quad\right] + \left[\frac{K - Nwt}{Krt}\frac{dw}{w}\right]$$

Here the terms are grouped by the square brackets into two columns. The first column shows the component of a change in the value of capital due to "productive" features (more labour, a longer period of production, greater output). The second column shows the component due to "financial" features (changes in the wage and interest rates). It is easy to verify that the different versions in each column are vertically equivalent: i.e. the "productive" component and the "financial" component of a change in K are respectively the same, whether considered (a) in terms of cost or (b) in terms of capitalization, and in the case of the "financial" element whether ascribed (c) to an interest change or (d) to a wage change. Let us make this distinction explicit by defining two synthetic variable, k and p, with the following properties at a certain equilibrium point:

$$(7) \quad kp = K$$

$$(8a)\ \frac{dk}{k} = \frac{dN}{N} + \frac{1 - e^{-rt}}{rt}\frac{dt}{t} \qquad (8b)\ \frac{dp}{p} = \frac{dw}{w} + \left(\frac{rt}{1 - e^{-rt}} - 1\right)\frac{dr}{r}$$

In these definitions, K is broken into two components which may be interpreted as a "quantity" k, and a "price" (in terms of product) p; $\frac{dk}{k}$ is identified with the first column of (6), and $\frac{dp}{p}$ with the second column. The stated properties cannot hold generally, because the product of the integrals of (8a) and (8b) is not, in general, K. Nevertheless the definitions involve no contradiction if they are restricted to a particular set of equilibrium values of w, r, and t—namely, those prevailing at the equilibrium point for which in any particular case the differentials in (5) or (6) are also calculated. We shall return in a moment to the question of integrating (8a) and (8b) so as to define k and p for other points.

Next we can put $dK = K\left(\frac{dk}{k} + \frac{dp}{p}\right) = pdk + kdp$ in (5a), and arrange the result as follows:

$$(5a') \quad \frac{dQ}{pdk} = r + \frac{kr}{dk}\left(\frac{dp}{p} - \frac{dw}{w} - \left(\frac{rt}{1 - e^{-rt}} - 1\right)\frac{dr}{r}\right)$$

According to the definition of $\frac{dp}{p}$ in (8b), the second term—which previously showed the Wicksell effect—is now identically zero: when the increment of capital is taken as $K\,\frac{dk}{k}$, or pdk ("the value of the change"), its marginal productivity corresponds with the rate of interest. The same result can of course be obtained by a similar substitution in the other three versions of (5). Here is the common feature of all four versions of the Wicksell effect, that the effect disappears when the marginal change in capital is measured so as to exclude $K\,\frac{dp}{p}$, or kdp, which is the revaluation of the capital stock resulting from an associated marginal change in wage and interest rates. *The Wicksell effect is nothing but an inventory revaluation.*

The wage rate, previously given by (2) as the average product of labour discounted over the period of production, may now also be derived as the marginal productivity of labour. Differentiating (1), using (3), and substituting for dt from (8a), we obtain:

$$(9)\quad \frac{dQ}{Q} = e^{-rt}\frac{dN}{N} + (1 - e^{-rt})\frac{dk}{k}$$

By (1), (2), and (4b), the two coefficients in (9) are the proportional shares of wages and profits in output, $\frac{Nw}{Q}$ and $\frac{Kr}{Q}$, respectively. Therefore:

$$(10)\quad dQ = wdN + rpdk$$

Accordingly, when the quantity of capital k is held constant, $w = \frac{dQ}{dN}$.

The component pdk is the value at ruling prices (in terms of product) of an increment of capital goods, and so corresponds with the usual idea of investment, saving, or accumulation. It may be convenient to call $\frac{dQ}{pdk}$ $(= r)$ the *marginal efficiency of investment*, reserving the term *marginal productivity of capital* for $\frac{dQ}{dk}$ $(= rp)$. In relation to our earlier discussion, dk is an increment of capital measured "in terms of its own technical unit" (like meccano sets), while pdk is an increment measured in terms of "an equilibrium dollar's worth".

But we are now a step forward. The definitions of k and p in (7) and (8) can be recognized in their essential character as the differential definitions of *chain indexes* of quality and price, by which Divisia provided " an elegant logical justification" of Marshall's original invention of the chain index.[24] Thus although (8a) and (8b) cannot

24. Ragnar Frisch, *Econometrica* 1936, pp. 7-8. The elementary index-number formula used to construct each link of the chain will vary, as Frisch points out, "according as we choose the approximation principle for the steps of the numerical integration".

usually be integrated to give exact measures of k and p, such that $kp = K$ at every point, they can in principle be integrated numerically in successive small "links" (correcting the weights as each link is added) so as to form a consistent pair of chain indexes of the "quantity" and "price" of capital. With these indexes approximate structural comparisons "in the large" between different equilibrium situations may be made. The index k enters with N in the production function, while the index p measures for each point of the production function the amount of accumulation in terms of product necessary to achieve a given addition to k—in effect, converting "an equilibrium dollar's worth" at one point into its productive equivalent at another point.

This operation can most easily be visualized by considering a special case in which (8a) and (8b) lend themselves to exact integration—namely, the case in which the function $f(t)$ is of constant elasticity, and may be written $f(t) = t^{\alpha}$. Then by (3) $rt = \alpha$. The proportional share of profits in output is also now a constant, which it is convenient to write $1 - e^{-rt} = \beta$. Therefore (8a) and (8b) become

$$(8a') \ \frac{dk}{k} = \frac{dN}{N} + \frac{\alpha}{\beta}\frac{dt}{t} \qquad (8b') \ \frac{dp}{p} = \frac{dw}{w} + (\frac{\alpha}{\beta} - 1)\frac{dr}{r}$$

and in this form they give immediately the integrals

$$(a') \ \ k = C_1 N t^{\frac{\alpha}{\beta}} \qquad (b') \ \ p = C_2 w r^{\left(\frac{\alpha}{\beta} - 1\right)}$$

where C_1 and C_2 are constants of integration.[25] The production function (1) may now be expressed in terms of N and k, and its partial derivatives with respect to these factors of production will appear as w and rp:

$$(1') \quad Q = N^{1-\beta} k^{\beta} \qquad (2') \quad w = (1 - \beta)\frac{Q}{N} \qquad (3') \quad rp = \beta \frac{Q}{K}$$

Given the definitions of k and p in (a′) and (b′), the new system is in all respects the equivalent of Wicksell's, as the reader may readily confirm by substitution. Although the wage rate w and the yield (or quasi-rent) of a unit of capital rp are derived from the new production function as the marginal productivities of labour and capital, it seems at first sight that in order to discover r we must know p, and *vice-versa*. However, in (b′) there is another relation between

25 In order that k and p may satisfy (7) and (4), C_1 and C_2 must satisfy

$$C_1 C_2 = (e^{\alpha} - 1) \alpha^{-\frac{\alpha}{\beta}}$$

It is convenient to choose units so that $C_1 = 1$. This choice accounts for the absence from (1′) below of any explicit constant of integration.

w, r, and p, which enables r and p to be separately determined once the values of w and rp are given at any point of the production function.[26]

When the elasticity of $f(t)$ is not constant, this exact formulation in terms of k and p is no longer possible "in the large". Nevertheless the chain indexes of k and p are available as approximate measures, and they will play in principle the same role as k and p in the special case just considered.[27]

But why bother to show that with the help of chain indexes the neo-classical scheme can approximately mimic the solution of a highly artificial problem already obtained in an exact form by Wicksell? One answer is that Wicksell's analysis is exact only when K—the *value* of capital in terms of product—is taken as an independent variable. To consider the effect of a given amount of *accumulation*—the forgone consumption of a given amount of product—Wicksell too would have been driven to approximations and index-numbers. Another answer is that the elements which appear in our definitions of the indexes k and p are merely particular illustrations, drawn from Wicksell's model, of the "productive" and the "financial" attributes of capital goods that have to be distinguished in measuring their "quantity" and "price": index-number measurements may still be appropriate when capital does not take those particular forms which enabled Wicksell to specify its productive effect directly in terms of a period of time.

Wicksell himself thought of the period of production or period of investment as no more than a notional index of the time-aspect of capital—"a mathematical concept, without direct physical or psychic significance", but which "should, nevertheless, be retained as a concise general principle, reflecting the essence of productive capital".[28] If Joan Robinson will allow Wicksell in this spirit to draw a production function involving N, t, and (indirectly) K, she ought not to object if others prefer to draw one involving N, k, and (indirectly) p: for there is, as we have seen, a method by which one scheme may be translated into the other.

26. In this special case where the production function is such that the proportional share of each factor in output is constant, there is obviously no difficulty in extending the above analysis to cover any number of different factors of production. As far as I can see, the chain index approach in the general case also extends to any number of factors, provided that continuous adjustments in factor proportions are assumed to be possible. Champernowne (*R.E.S.* 1953-54, pp. 121-125, 132-135) shows that the chain index in general breaks down for more than two factors when techniques are discontinuous.

27. Of course a chain index is not necessarily a "better" approximation than some other kind of index-number. For the present purpose, however, the chain index in its Divisia formulation is very convenient, in that it shows a consistent way of making approximate measurements "in the large", while keeping the advantage of theoretical exactness "in the small".

28. *Lectures*, p. 184.

III. *Åkerman's Problem*

By the same method, Gustaf Åkerman's problem of durable capital equipment—as analysed by Wicksell in a celebrated essay[29]—can also be solved in accordance with the marginal productivity theory. Wicksell's analysis was mainly intended to refute Åkerman's claim that this could be done.

In the model which Wicksell developed for the purpose, capital consists of axes, which can be made more or less durable by putting more or less labour into their manufacture; the optimum life of an axe, n years, is chosen to maximize profits; the stock of axes in the stationary equilibrium is a "balanced equipment" with a uniform age distribution from o to n years; M labourers out of the total labour force A are occupied in replacing the nth part of the stock that wears out each year, while A-M "free labourers" co-operate with the stock of axes to produce a (net) output π. K is the value of the stock of axes (in terms of product), l the wage rate, and ρ the rate of interest.[30]

K is evaluated by Wicksell in equation (15) p. 283. With one substitution from equation (4) p. 276, (15) becomes:

$$(15.1) \quad K = Mnl\left(\frac{1}{1 - e^{-\rho n}} - \frac{1}{\rho n}\right)$$

Here Mnl is the replacement cost of the whole stock of axes, while the bracketed expression can be recognized as the Champernowne-Kahn formula for the value of a "balanced equipment" as a proportion of its replacement cost.[31] Differentiating (15.1) as it stands, and

29. First published in Swedish in 1923, then republished in 1934 with the English edition of *Lectures*, Vol. 1, pp. 274-99). Until the Joan Robinson—Kahn-Champernowne papers of 1953-54, this essay seems to have been the only analysis available in English of the specific questions posed for (long-run) capital theory by durable, depreciable, capital equipment.

30. Wicksell's notation is preserved. In this case no attempt will be made to re-formulate Wicksell's model. Assuming that the interested reader will look up the original, we give the essentials of the argument with a minimum of incidental explanation.

31. Wicksell's derivation of (15) is explained by R. G. D. Allen, *Mathematical Analysis for Economists*, p. 405. The Champernowne-Kahn formula is derived by Champernowne and Kahn in four different ways (*R.E.S.* 1953-54, pp. 107-111). Joan Robinson reports in her preface that C. A. Blyth has derived it independently.

The underlying principle can be seen in graphical terms. The cost or value of a "machine" is equal to its future gross earnings discounted to the present moment. Given the prospective earnings at each point of its life, and given the rate of interest at which they are to be discounted, a curve showing the machine's value as a function of its age will fall from its starting-point at age o (replaceplacement cost) down to zero at age n years when it falls to pieces. The average value of the machine *per year of life* is the area under the curve divided by n. A "balanced equipment" of such machines is of uniform age distribution from o to n, and so repeats in cross-section the life-history of a single machine. The average value *per machine* in a "balanced equipment" is therefore also the area under the curve divided by n. In the particular case when the earnings of a machine are at a constant rate throughout its life, the Champernowne-Kahn formula gives the ratio of this average value to the original value at age o.

then making a substitution from Wicksell's equation (9) p. 278, we obtain the logarithmic total differential of K:

$$(15.2)\ \frac{dK}{K} = \left[\frac{dM}{M} + \frac{(1-\nu)\ \ (\nu+\phi(\nu))}{\nu+\phi(\nu)-1}\frac{dn}{n}\right] + \left[\frac{dl}{l} + \left(\frac{1-\nu}{\nu+\phi(\nu)-1} - \nu\right)\frac{d\rho}{\rho}\right]$$

The proportional change in the value of capital is split by the square brackets into a "productive" and a "financial" component, just as in (6a) of Part II. Again we identfy $\frac{dk}{k}$ with the first component, and $\frac{dp}{p}$ with the second. This time the distinction is easier to visualize. The "technical unit" of capital in which k is measured is in effect a *standard axe* (of given durability and age), while p is the value of such an axe, calculated at current wage and interest rates. This follows simply from the fact that the second component of (15.2) is the differential of (15.1) with respect to l and ρ, calculated as of constant M and n. In the present model the definition of a "standard axe" creates no index-number problems: Wicksell's constant elasticity formulae mean that the coefficients in (15.2) are constants, so that k and p can be obtained by direct integration as indexes with correct and constant weights at every point. Moreover, M is a constant proportion of the total labour force A (Wicksell, p. 287), and is therefore determined when A is taken as an independent variable.

The rest follows as in Part II—the Wicksell effect disappears, the production function[32] can be written in terms of A and k, etc. In fact, with an appropriate revision or re-interpretation of the various constants, our earlier equations (a′), (b′), (1′), (2′), and (3′) will now serve as an exact representation in neo-classical form of Wicksell's analysis of Åkerman's problem.

IV. *The Wicksell Effect in Reverse*

One new feature emerges. In the model of Part II increasing K (or k) always means increasing p: the wage rate rises and the interest rate falls, but the net effect is necessarily a rise in the value of a unit of capital in terms of product. Thus the Wicksell effect is an apparent absorption of capital. However, in the model of Part III it turns out that the two components of (15.2) may very well be of

32. With k defined as above, it can be shown that the production function given by Wicksell in equation (17 *bis*) p. 287 is correctly reproduced by the integral of the following expression:

$$\frac{d\pi}{\pi} = \left(1 - \beta\frac{\nu+\phi(\nu)-1}{\nu+\phi(\nu)}\right)\frac{dA}{A} + \beta\,\frac{\nu+\phi(\nu)-1}{\nu+\phi(\nu)}\frac{dk}{k}$$

By Wicksell's assumptions, the coefficients are constants.

opposite sign. So long as the "convexity" conditions for profit maximization are satisfied, a higher wage rate and a lower interest rate must still accompany increasing K (or k), but the interest effect on p may now outweigh the wage effect.[33] The value of a "standard axe" may fall. ***In this event the Wicksell effect goes into reverse.***

When Wicksell calculated $\frac{d\pi}{dK}$ he found again that by this measure Åkerman's and von Thünen's thesis was "not verified", but he found also that in his new model $\frac{d\pi}{dK}$ might actually exceed the rate of interest—i.e. he discovered the Wicksell effect in reverse. This phenomenon left Wicksell very puzzled, and caused him to admit that his previous explanation, in terms of the absorption of capital in increased wages, was "*not* general".[34]

Once it is realized that the Wicksell effect merely reflects a revaluation of the capital stock, it is no longer puzzling that it may go in either direction. When wages rise and interest falls, whether the value of a "standard axe" goes up or down in terms of product may be expected to turn (broadly speaking) on a comparison of the relative importance of the two factors for the axe on the one hand, and for the product on the other. In general, there is no presumption either way. But in Wicksell's previous models, before his analysis of Åkerman's problem, the product typically emerged only at the last and most "capitalistic" stage of production. In such models (as in Part II above) a higher wage rate and a lower interest rate must depress the final product, and elevate the goods-in-process at the earlier stages, in relative value. Hence Wicksell's surprise on finding himself at the age of 72 in a new world of durable capital equipment, in which this rule no longer applies.

33. Here we are looking at capital from the viewpoint of cost, as in (6a). It is possible as before to express $\frac{dp}{p}$ in terms of either the wage rate or the interest rate alone. For instance, corresponding with (6c), the second component of (15.2) may be written:

$$\frac{dp}{p} = -\left[\nu + \beta(1-\nu) - \frac{1-\nu}{\nu + \phi(\nu) - 1}\right]\frac{d\rho}{\rho}$$

The second-order maximization conditions imply that β and ν are each less than unity, and that the denominator of the third term is positive. But the sign of the sum within the brackets (which in (6c) is always positive) depends on the relative magnitudes of β and ν; it can be negative if β is small and ν neither very near unity nor very near zero.

34. pp. 292-293. It is interesting to note that Wicksell in these pages experimented with the possibility of adjusting his measure of the increase in capital, by deducting the effect of the rise in the wage rate, precisely as we have done in defining $\frac{dk}{k}$ and $\frac{dp}{p}$. He failed to reach the same conclusion only because he did not allow for the lower interest rate as well as for the higher wage rate.

What is more puzzling is why Joan Robinson thirty years later should write as if she and Wicksell were both back in the old world where capital was goods-in-process. Her rule that "at a lower wage rate there is a smaller value of a given type of machine" need not hold even for Wicksell's hand-made axe, far less for a typical machine, itself a capitalistic product.[35] The revaluation of a given machine in an opposite direction to the wage rate (in the same direction as the interest rate) is a reverse Wicksell effect, but there is nothing perverse about it[36], and in general it is just as likely to happen as its obverse, the original Wicksell effect.

Most puzzling of all is how the possibility of a shift in relative value between capital good and product *in an unpredictable direction* can become in Joan Robinson's hands "the key to the whole theory of accumulation and of the determination of wages and profits."

35. The influence of interest on the value of an axe is confined in (15.1) and (15.2) to the Champernowne-Kahn term of (15.1)—i.e. to the effect of the interest rate on the value of a "balanced equipment" of axes as a proportion of its replacement cost. The latter consists of labour cost alone. If axes were themselves made with the co-operation of capital, their replacement cost would also contain an interest element, and it would be much easier for the reverse Wicksell effect to occur.

36. The perverse case discussed by Joan Robinson, in which a higher wage rate and a lower interest rate make a *less* mechanized technique relatively profitable, has nothing to do with the direction of the Wicksell effect, though one might easily get the impression that Joan Robinson thinks it does (see *R.E.S.* 1953-54 pp. 95-96, 106, and *The Accumulation of Capital*, pp. 109-110, 147-148, 418). The perversity arises essentially from a failure over a certain range of the second-order ("convexity") conditions for profit maximization, as indeed Wicksell pointed out in his analysis of Åkerman's problem (pp. 294-297, especially the footnote on p. 295). Only in his earlier goods-in-process model would a reverse Wicksell effect imply the failure of those conditions, and so perversity.