5.6 & 5.7

Here are the number groups we have discussed so far and their relationships:

All Real numbers have a decimal equivalent. That is, each real number can be written as a unique decimal. These decimal equivalents can be grouped into several different kinds.

1. **Terminating**: this means the decimal ends. For example, .25, 3.7456, or .000058.

2. **Repeating**: this means that after a certain point the digits repeat forever. We often use a bar over the repeating digits.
   For example, $.3\overline{3}$, $.123\overline{123}$, $.23\overline{7}$

3. **Non-terminating & non-repeating**: this means that even if it looks like there is a pattern, there is none. The decimal equivalent never ends and never repeats.

   For example, 1.01001000100001…, 2.012345678910111213141516…

Changing fractions to decimals Divide the numerator by the denominator.

$\frac{3}{8}$

$\frac{5}{9}$

$\frac{59}{10}$

Changing decimals to fractions

Terminating decimals Read the number, reduce the fraction

0.124

0.3497

Repeating decimals Set it equal to *N*
Multiply both sides by a power of 10
Subtract both equations
Solve for *N* and reduce the fraction

$.777\overline{7}$

$.\overline{124}$

Properties of the Real Numbers.

Closure of Addition	Closure of Multiplication
Associative of Addition	Associative of Multiplication
Commutative of Addition	Commutative of Multiplication
Addition Identity	Multiplicative Identity
Additive Inverse	Multiplicative Inverse

Distributive Property of Multiplication over Addition

Algebraic Structure:

When you hear the word "algebra," you probably think of a class from high school. However, mathematicians use the word "algebra" to describe a generalization of arithmetic, that is a structure with a set of symbols that operate according to certain properties.

We can take some of these properties and together with a set of elements and an operation, we can form a Group.

A **Group** must have: a set of elements, an operation, and these properties must hold true.
- Closure under the operation
- Associative of the operation
- Identity
- Inverse

Example: the set of Integers and the operation of addition Group?

- Closure under Addition in the Integers?
- Associative of Addition?
- Additive Identity?
- Additive Inverse?

Example: the set {– 1, 0, 1} and the operation of multiplication Group?

- Closure under multiplication?
- Associative of multiplication?
- Multiplicative Identity?
- Multiplicative Inverse?

Example: the set { – 1, 0, 1} and the operation of addition Group?

Closure under addition?

Associative of addition?

Additive Identity?

Additive Inverse?

Example: the set of Natural Numbers and multiplication Group?

Closure under multiplication?

Associative under multiplication?

Multiplicative Identity?

Multiplicative Inverse?

A **Field** must have a set, 2 operations and these properties must hold true:

- Closure under both operations
- Associative of both operations
- Identity of both operations
- Inverse of both operations
- **Commutative** of both operations
- **Distributive** property of one operation over the other.

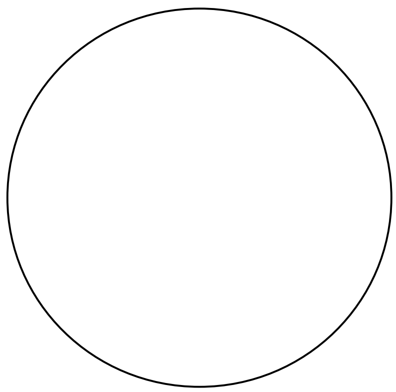
Notice that a Field needs two operations and two more properties than a Group.

Are the Real Numbers a Field?

Clock Arithmetic

Consider a mathematical system based on a 12-hour clock. Here is arithmetic done with a finite set. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

It is closed for addition since adding any two numbers yields a sum in the set.



We can see that sometimes the same time has different names. For instance 2 and 14 are the same place on the clock. So are 6 and 18.

Find the following sums:

7 + 5	4 + 11	9 + 9	12 + 3

We can also see that subtracting, multiplying, and dividing can be defined on this set:

Subtraction: $a - b = x$ means that $a = b + x$

Multiplication: $a \times b = ab$ means to add b to itself a times

Division: $a \div b = x$ $\left(\text{or } \frac{a}{b} = x\right)$ means $a = bx$, provided that b has an inverse for multiplication.

Find the following:

$4 - 9$

4×9

$4 \div 7$

$4 \div 9$

Instead of calling this "clock arithmetic" and limiting ourselves to 12 elements in the set, it is called "modulo 12 arithmetic" or "mod 12".

We know that 2 and 14 share the same place on the clock. We say that 2 (mod 12) and 14 (mod 12) are **congruent**. We write it as $2 \equiv 14 \pmod{12}$.

Definition: Real numbers *a* and *b* are **congruent modulo *m*,** written as $a \equiv b \pmod{m}$, if *a* and *b* differ by a multiple of *m.* That is, if $m \mid (a - b)$.

Are the following true?

$3 \equiv 8 \pmod{5}$

$3 \equiv 51 \pmod{8}$

$3 \equiv 19 \pmod{9}$

Modulo arithmetic is sometimes called Remainder arithmetic since you can check the remainders when dividing by the mod. Congruent numbers will have the same remainder when divided by the mod.

$17 \equiv 28$ mod 11 since 17 divided by 11 has a remainder of 6 and
26 divided by 11 has a remainder of 6

Solve each equation for x:

$4 + 9 \equiv x \pmod{5}$

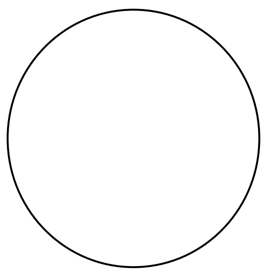
$2 + 4 \equiv x \pmod{5}$

$2 - 4 \equiv x \pmod{7}$

$7 \times 5 \equiv x \pmod{7}$

$3 - 5 \equiv x \pmod{12}$

Is the set {0, 1, 2, 3, 4} and the operation of addition (mod 5) a Group?



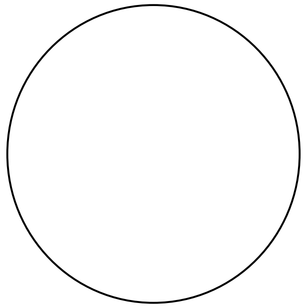
Closure for addition?

Associative for addition?

Identity for addition?

Inverse for addition?

Is the set {0, 1, 2, 3} and the operation of multiplying (mod 4) a Group?



Closure of multiplication?

Associative of multiplication?

Identity for multiplication?

Inverse for multiplication?