Math 1   5.2   Prime Numbers

Before we talk about prime numbers, we will spend some time with divisibility….

Definition: For whole numbers A and D, with $D \neq 0$, if there is a whole number Q such
               that $A = D \cdot Q$, then we say that   D divides A
                                                          D is a factor of A
                                                          D is a divisor of A
                                            and   A is a multiple of D.

               We denote this relationship by $D \mid A$

Example: $5 \mid 35$ because there is _____

Example: Does $5 \mid 12$?

Some rules of Divisibility:

1

2

3

4

5

6

8

9

10

12

Along with divisibility is the idea of divisors of a number.

What numbers divide….

    12?

    63?

    104?

    17?

The numbers that divide a natural numbers are called **divisors** and also called **factors.**

When a natural number has exactly two distinct divisors, we say that number is **Prime**.

A natural number that has more than 2 distinct divisors is called **Composite.**

Is 1 a prime number?

How can we find prime numbers?

Sieve of Eratosthenes: (a Greek mathematician who lived in 200 BC)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

How can you know you have crossed out enough numbers and that the rest are prime?

Proof: Assume n is composite. Suppose the factors are a and b. So, $n = a \cdot b$

Both a and b cannot be greater than $\sqrt{n}$ because, if they were both greater than $\sqrt{n}$, then $ab > \sqrt{n} \cdot \sqrt{n}$. But $\sqrt{n} \cdot \sqrt{n} = n$. This would mean that $ab > n$. This is called a contradiction since we said $ab = n$. So, one of the factor of n must be less than $\sqrt{n}$.

In our Sieve we looked at the numbers through 100. $\sqrt{100} = 10$ So, we only have to use the Sieve through the closed prime number to 10, which is…

So, from our Sieve of Eratosthenes, the prime numbers between 1 and 100 are:

Prime Factoring:

Consider the number 18.   What are all the factors (divisors) of 18?

Notice the relationship between the pairs of factors.

We can reduce the numbers of factorizations to a unique factorization.

This unique factorization is so important it is called:

The Fundamental Theorem of Arithmetic. It says that every natural number greater than 1 is either prime or is a product of primes and that prime factorization is unique to that natural number (except for the order of the factors).

Finding the prime factorization of a number:

Ladders and Trees.

385   1400

Examples for you to try:

3465   4752

Canonical form: means to list the prime factors in order even if a prime is not represented.

For example: $140 = 2^2 \cdot 5 \cdot 7$ in canonical form, it would be: $140 = 2^2 \cdot 3^0 \cdot 5 \cdot 7$

Greatest Common Factor: the largest number that is a divisor (factor) of two or more natural numbers.

Ways to find the GCF:

    1.   Find the prime factors of the numbers.

    2.   Fill in a Venn Diagram. The intersection is the GCF.

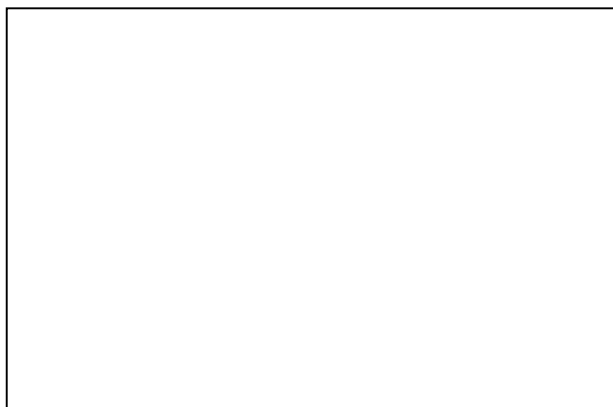   3.   OR   take the smallest exponent of the common factors.

Example: Find the GCF of 3150, 588, and 280

    1.   3150 =

             588 =

             280 =

    2.   Draw the Venn Diagram

OR   3.   Take the smallest exponent of the common factors from 1.

Least Common Multiple: The smallest number that is a multiple of two or more natural numbers.

Ways to find the LCM:

From the prime factorization of the numbers, take the largest exponent of each of the factors.

From the Venn Diagram take all the factors in the union of the sets.

From the example above the LCM of 3150, 588, and 280 is _____

Now you try: Find the GCF and LCM of 2625 and 4950

We can use the LCM to find the answers to certain word problems. For example…

At the party store, paper plates come in packages of 30, paper cups come in packages of 15, and napkins come in packages of 20. what is the least number of plates, cups, and napkins that can be purchased so that there is an equal number of each?

Finding Prime Numbers…

Since the time of Eratosthenes mathematicians have been trying to find prime numbers and develop formulas to find prime numbers. In recent history finding primes has become necessary because they are the basis of encrypting the information sent between computers—when you order online, check your bank balance, log in to sites like WebAssign, etc.,… basically, anytime you are on a https: site.

An early contender as a prime number finder was: $n^2 - n + 41$

    n = 1 $n^2 - n + 41 =$

    n = 2 $n^2 - n + 41 =$

    n = 3 $n^2 - n + 41 =$

    n = 4 $n^2 - n + 41 =$

    n = 41 $n^2 - n + 41 =$

Another serious attempt was made by Fermat in the 1600s. It was: $2^{2^n} + 1$

    n = 1 $2^{2^n} + 1 =$

    n = 2 $2^{2^n} + 1 =$

    n = 3 $2^{2^n} + 1 =$

    n = 4 $2^{2^n} + 1 =$

    n = 5 $2^{2^n} + 1 = 4,294,967,297$ which turns out to be composite!
                                            It divides by 641, which is prime.

A French priest, Mersenne, in 1644, discovered that $2^{257} - 1$ is a prime number.

In 1970 a Russian named Matyasevich discovered several specific polynomials that generate only prime numbers but they are quite complicated.

The current largest prime number was discovered in 2009 using the GIMPS project. It is $2^{43,112,609} - 1$. It has 12 million digits!

The GIMPS project (Great Internet Mersenne Prime Search) uses thousands of small computers to look for and check primes. People sign up to let GIMPS use their computers (usually during the night) and by linking together all these computers, they have more computing power than the supercomputers.

You can find more information on the GIMPS project and sign up to help by lending computer time on your computer by going to: www.mersenne.org/prime.htm

There is no largest prime number. In fact there are infinitely many primes. So the search continues…

Goldbach's Conjecture: In 1742 Christian Goldbach observed that every even number greater than 2 can be expressed as the sum of two prime numbers. It is called a conjecture because even though it seems to be true, no mathematician has been able to <u>prove</u> that it is true.

Examples: 4 = 2 + 2

6 =

18 =

60 =

70 =

Perfect Numbers: a natural number that is equal to the sum of all its proper divisors. (Proper divisors are less than the number)

Examples: 6 has divisors 1, 2, 3, and 6. The proper divisors of 6 are 1, 2, and 3

6 = 1 + 2 + 3   So, 6 is a perfect number.

8 has proper divisors of:

and the sum of those divisors:

24 has proper divisors of:

and the sum of those divisors:

28 has proper divisors of:

and the sum of those divisors:

It turns out that every perfect number can be written in the form:

$$2^{n-1}\left(2^n - 1\right)$$

Let's check 6:   if n = 2, then we have $2^{2-1}\left(2^2 - 1\right)$ = 2 (4 − 1) = 6

Now check _____ :

In addition to Perfect Numbers there are also Friendly Numbers, Abundant Numbers, and Deficient Numbers. All based on the relationship of the sum of their proper divisors to the number.