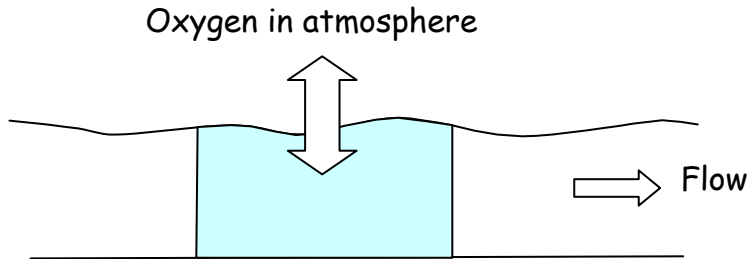


Derivation of the Dissolved Oxygen Sag Equation

Consider an element of water in a stream. We will track the element as it moves downstream with the general flow of water.



Setting up the differential equation

The mass balance equation written out in words is:

$$\text{Accumulation} = \text{In} - \text{Out} + \text{Reaction}$$

Because we are looking at the element by itself, there is no flow of water into or out of it. So the mass balance on oxygen becomes:

$$V \frac{dC}{dt} = 0 \text{ in} - 0 \text{ out} + (\text{Rate of } O_2 \text{ removal by biodegradation})V + (\text{Rate } O_2 \text{ addition from the atmosphere})V$$

Let's look at these two rates. The rate of O_2 removal by biodegradation is:

$$\left(\frac{dC}{dt} \right)_d = (\text{Rate of BOD decay}) = -k_d L$$

where

L = ultimate BOD of organics remaining in the water at any time t

k_d = first-order degradation rate constant. k_d is related to, but is not numerically the same as, k_{BOD} , the rate constant in the BOD test.

The rate of O_2 transfer from the atmosphere (called reaeration) is:

$$\left(\frac{dC}{dt}\right)_r = k_r(C_s - C)$$

where

C = dissolved oxygen concentration (DO)

C_s = dissolved oxygen concentration at saturation, predicted from Henry's Law

k_r = reaeration rate constant. This value will be related to the characteristics of the stream.

Note that this is the same form of equation we used in the gas transfer lab. Now, substitute the rate expressions into the mass balance.

$$V \frac{dC}{dt} = \left(\frac{dC}{dt}\right)_d V + \left(\frac{dC}{dt}\right)_r V$$

$$\frac{dC}{dt} = -k_d L + k_r(C_s - C)$$

Mathematically, it is convenient to express everything in terms of the deficit, D.

$$D = (C_s - C)$$

$$\frac{dD}{dt} = \left(0 - \frac{dC}{dt}\right) = k_d L - k_r(C_s - C)$$

This is the basic differential equation that must be solved.

Solving the differential equation

Solving the differential equation requires the use of integrating factors. First substitute the BOD decay equation for L.

$$\frac{dD}{dt} = k_d L_0 e^{-k_d t} - k_r D$$

$$\frac{dD}{dt} + k_r D = k_d L_0 e^{-k_d t}$$

This arrangement matches the standard form:

$$\frac{dz}{dx} + Pz = Q$$

So, use the integrating factor:

$$e^{\int k_r dt} = e^{k_r t}$$

$$\frac{dD}{dt} (e^{k_r t}) + k_r D (e^{k_r t}) = k_d L_0 e^{-k_d t} (e^{k_r t})$$

$$\frac{d}{dt} (D e^{k_r t}) = k_d L_0 e^{(k_r - k_d)t}$$

Separate variables and integrate.

$$\int d(D e^{k_r t}) = \int (k_d L_0 e^{(k_r - k_d)t}) dt$$

$$D e^{k_r t} = \frac{k_d L_0}{k_r - k_d} e^{(k_r - k_d)t} + C$$

Apply the boundary condition that at $t = 0$, $D = D_0$

$$D_0 e^0 = \left(\frac{k_d L_0}{k_r - k_d} \right) e^0 + C$$

$$C = D_0 - \left(\frac{k_d L_0}{k_r - k_d} \right)$$

$$D e^{k_r t} = \left(\frac{k_d L_0}{k_r - k_d} \right) e^{(k_r - k_d)t} + D_0 - \left(\frac{k_d L_0}{k_r - k_d} \right)$$

$$D = \left(\frac{k_d L_0}{k_r - k_d} \right) e^{-k_d t} + D_0 e^{-k_r t} - \left(\frac{k_d L_0}{k_r - k_d} \right) e^{-k_r t}$$

Finally, after grouping like terms, the integrated equation is:

$$D = \left(\frac{k_d L_0}{(k_r - k_d)} \right) (e^{-k_d t} - e^{-k_r t}) + D_0 e^{-k_r t}$$

where

L_0 = ultimate BOD in the water at $t = 0$

D_0 = deficit at $t = 0$.

Note that both L_0 and D_0 are the values in the stream *after* any external waste streams have been mixed in.

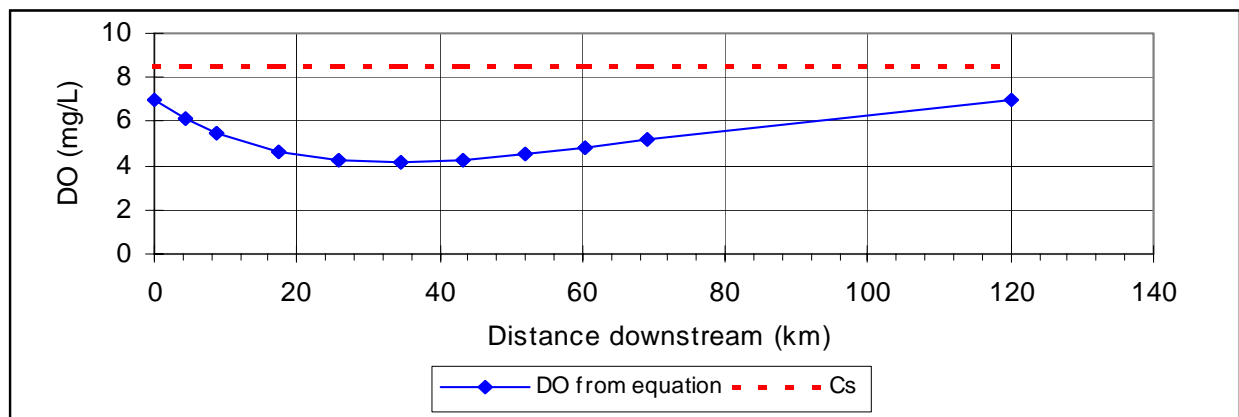
This equation is known as the DO sag equation (for its distinctive shape) or the Streeter-Phelps equation, after the gentlemen who first published it in 1925.

Using the DO Sag equation

Of course, the fish aren't interested in the deficit. They want to know the concentration. This can be calculated from:

$$C = (C_s - D) = C_s - \left(\left(\frac{k_d L_0}{(k_r - k_d)} \right) (e^{-k_d t} - e^{-k_r t}) + D_0 e^{-k_r t} \right)$$

When this equation is plotting this equation with typical values, the following curve is generated:



We can use the DO sag equation to find the concentration at *any* point downstream of a waste discharge as long as we know the travel time to that point, which is easy to calculate if the stream velocity is known.

To protect fish, engineers often want to know the minimum oxygen concentration that can be expected as a result of a waste discharge. This is called the *critical* point (critical from the point of view of the fish). The travel time to the lowest point in the sag is the critical time. Using the critical time (t_c), we can easily calculate the critical deficit (D_c), and the minimum DO concentration.

To find the critical time and deficit, differentiate the Streeter-Phelps equation and set it equal to zero.

$$\frac{dD}{dt} = 0 = \frac{d}{dt} \left[\left(\frac{k_d L_0}{k_r - k_d} \right) (e^{-k_d t} - e^{-k_r t}) + D_0 e^{-k_r t} \right]$$

$$0 = \left(\frac{k_d L_0}{k_r - k_d} \right) [-k_d e^{-k_d t} + k_r e^{-k_r t}] - D_0 k_r e^{-k_r t}$$

If you divide through by $e^{-k_r t}$, bring the right hand term to the other side of the equal sign, take the natural log of both sides, and solve for $t = t_c$, you get:

$$t_c = \left(\frac{1}{k_r - k_d} \right) \ln \left[\frac{k_r}{k_d} \left(1 - \frac{D_0 (k_r - k_d)}{k_d L_0} \right) \right]$$

To solve for the critical deficit, D_c , you could plug this expression for t_c into the Streeter-Phelps equation, but that would give you a massive algebra headache. It's easier to realize that at the bottom of the sag, $dD/dt = 0$. This allows us to go back to the differential equation.

$$\frac{dD}{dt} = k_d L_0 e^{-k_d t} - k_r D_c = 0$$

$$D_c = \left(\frac{k_d}{k_r} \right) L_0 e^{-k_d t}$$

Now, the minimum concentration C_{\min} can be calculated from: $C_{\min} = C_s - D_c$

Note that t_c and D_c are single values at a specific location. If we want the DO at any other location, we would use the Streeter-Phelps equation itself.